Quantum Jazz: Spectral Representation of Musical Harmony

Luca Iacovella, Ward Gauderis, Nicholas Harley, Lin de Huybrecht & Geraint A. Wiggins
Computational Creativity Lab — Vrije Universiteit Brussel

1. Musical background

Steedman (1984) introduced a **formal grammar** that describes a family of 12-bar blues chord sequences. The main musical structure captured by the grammar is the backward propagating authentic cadence.

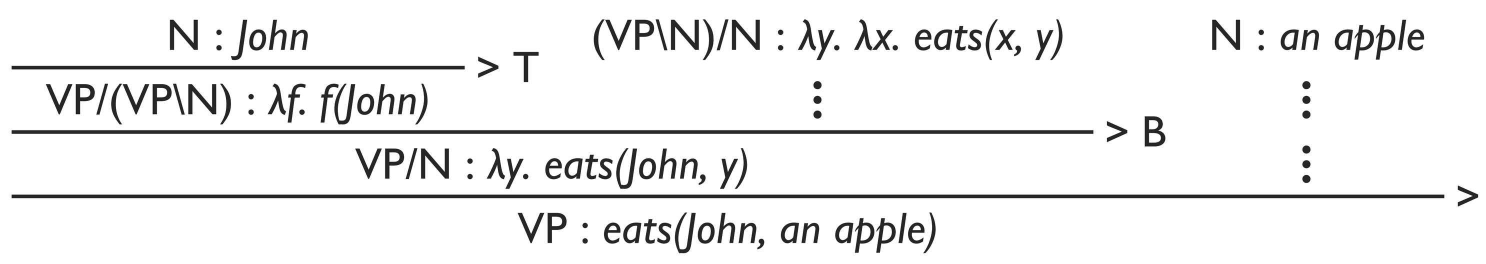
I	I	V^7	I
I	II^7	V^7	I
VI^7	II^7	V^7	I

The grammar was founded on the Longuet-Higgins Tonnetz (1962), a conceptual **tonal space**.

V^-	II^-	VI	III	VII	♯IV	$♯I^+$
$♭III^-$	$♭VII^-$	IV	I	V	II	VI^+
$♭I^-$	$♭V^-$	♭II	♭VI	♭III	♭VII	IV^+

2. Formal jazz grammar

Categorical grammars are formal grammars in which the meaning of a compound expression is derived from the meaning of its parts and its syntax—also called the **the principle of compositionality**. Combinatorial categorical grammars (CCGs) use combinators to combine syntactic categories in complex ways (Steedman, 1987).



The 1984 jazz grammar was redefined as a CCG to make it left-to-right parsable and to include semantics in terms of transformations on the Longuet-Higgins Tonnetz (Steedman, 2003). Non-exhaustive lexicon and cadence rule:

(1) $X(m) \coloneqq I_X(m)$ $: X$

(2) $X(m) \coloneqq V_X(m) \setminus V_X(m)$ $: \lambda x.\ x$

(3) $X^7 \coloneqq I_X^{\ 7} \setminus IV_X(m)^7$ $: \lambda x.\ leftonto(x)$

X $: origin$ $\Rightarrow$ $(I_X^{\ 7} \setminus I_X) \setminus (Y(m)^7 / X^7)$ $: \lambda cad.\ \lambda front.\ front + cad(origin)$

3. Compact closed category

Syntax and semantics will be defined as a **compact closed category** $(\mathcal{P}, \circ, \otimes, I, —^L, —^R)$:

- Objects $A, A^L, A^R, B, \ldots \in Ob(\mathcal{P})$
- Morphisms $f, g, \ldots \in Mor(\mathcal{P})$ between objects $f: A \to B$, $g: C \to D$
- Every object has an identity morphism: $1_A: A \to A$
- Composition operator $\circ$ is associative: $h \circ (g \circ f) = (h \circ g) \circ f$
- Monoidal tensor $\otimes$ is associative and has unit: $I \otimes A = A = A \otimes I$
- Monoidal tensor is structure-preserving: $1_A \otimes 1_B = 1_{A \otimes B}$ and $f \otimes g: A \otimes C \to B \otimes D$
- Objects have left and right adjoints: $A^L \otimes A = I$ and $A \otimes A^R = I$

4. Syntactic category

CCGs can be redefined as **pregroup grammars**, which are compact closed:

$N \quad (VP\backslash N)/N \quad N$ *becomes* $N \otimes (N^R \otimes VP \otimes N^L) \otimes N$

The syntactic category $\mathcal{J}$ is compact closed and based on Steedman's CCG (Iacovella, 2024):

- Chord objects $x_e^{\ q} \in Ob(\mathcal{J})$
  - Root $x \in \{C, D♭, D, E♭, E, \ldots\}$
  - Triad quality $q \in \{+, \Delta, –, \circ\}$
  - Extensions $e \subset \{7, ♭7, 11, ♭11, ♯11, \ldots\}$
  - Adjoints —, $—^L$ or $—^R$
- Transposition endofunctors $T^*_i : \mathcal{J} \to \mathcal{J}$
  - Defined for all intervals $i \in \{I, ♭II, II, ♭III, III, IV, \ldots\}$
  - Structure-preserving and only "modifies" root
- Subset of $Mor(\mathcal{J})$

(1) $x_\varnothing^{\ m} : I \to x_\varnothing^{\ m}$ $\qquad a : x_\varnothing^{\ m} \to x_{♭7}^{\ \Delta} \otimes y_{♭7}^{\ m\,R} \otimes x_\varnothing^{\ \Delta\,R} \otimes x_\varnothing^{\ \Delta}$

(2) $x_\varnothing^{\ m} : I \to T^*_V(x_\varnothing^{\ m\,R}) \otimes T^*_V(x_\varnothing^{\ m})$

(3) $x_{♭7}^{\ \Delta} : I \to x_{♭7}^{\ \Delta} \otimes T^*_{IV}(x_{♭7}^{\ m\,L})$ $\qquad (m \in \{\Delta, –\})$

5. Semantic category

The category of finite dimensional vector spaces is compact closed. The category for the semantics of the jazz grammar is defined as a **perceptual space FFreq** (Iacovella, 2024):

- Objects are spectral finite vector spaces V with basis frequencies spanning an octave (e.g., 440Hz to 880Hz with 1Hz increments) and values being amplitudes.
- Morphisms are linear transformations which represent a trajectory in spectral space V.
  - Rotation in spectral space $T : V \to V : x \mapsto Mx$

    with $M_{ij} = \begin{cases} 1 & \text{if } i = \lfloor j \sqrt[12]{2} \rfloor \\ 0 & \text{otherwise} \end{cases}$ (e.g., mod 440)
- f maps chord name to a spectral vector $f(x_e^{\ q}) \in V$
- s transforms scale degrees to semitones $s: \{I, ♭II, II, ♭III, III, \ldots\} \to \{0, 1, 2, 3, 4, \ldots\}$
- Functor between categories $F : \mathcal{J} \to FFreq : x_e^{\ q} \mapsto V$

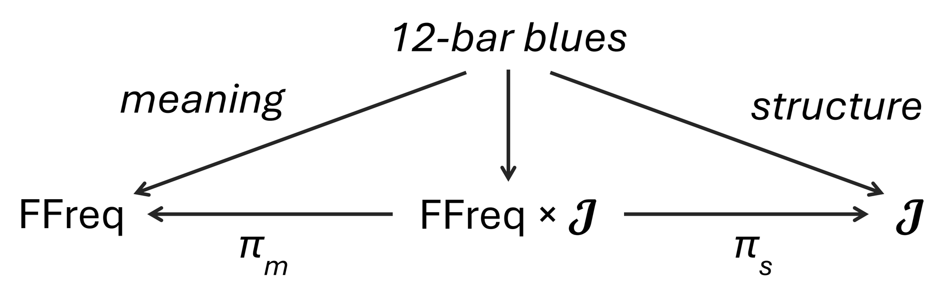
(1) $f(x_\varnothing^{\ \Delta}) \in V$

(2) $T^{s(I)} \in V \otimes V$

(3) $T^{s(IV)} \in V \otimes V$

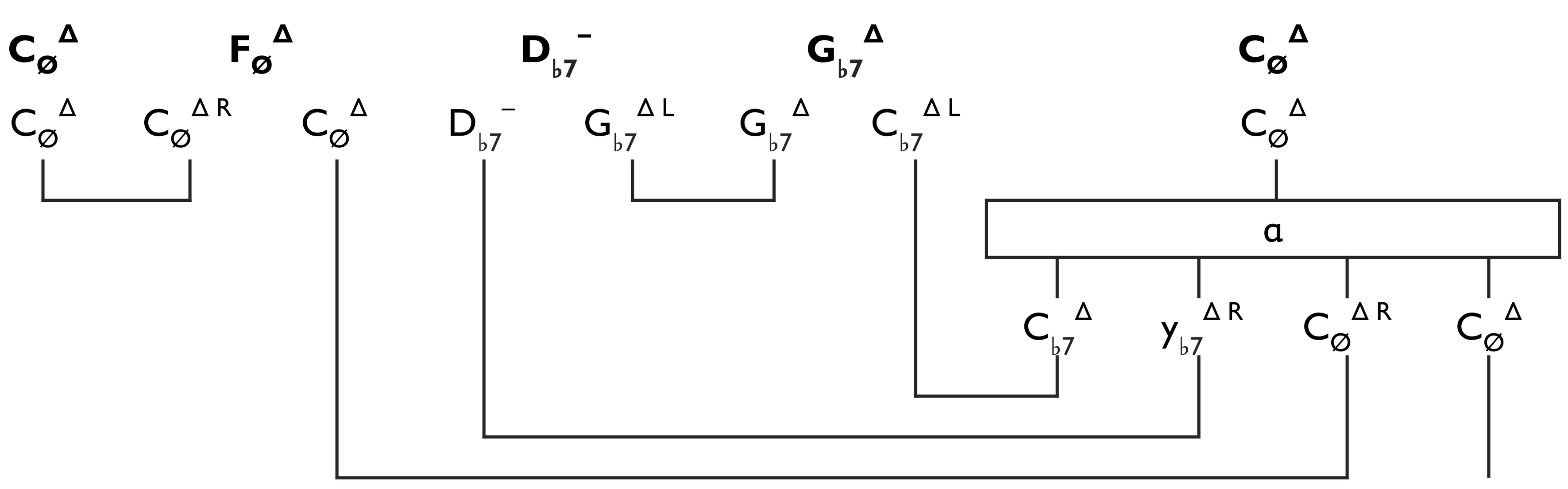
6. DisCoCat framework

DisCoCat takes a category-theoretical approach to unify the principle of compositionality and distributional semantics (Coecke et al., 2010). It allows for defining the syntactic category $\mathcal{J}$ and the semantic category FFreq in a **joint category**.



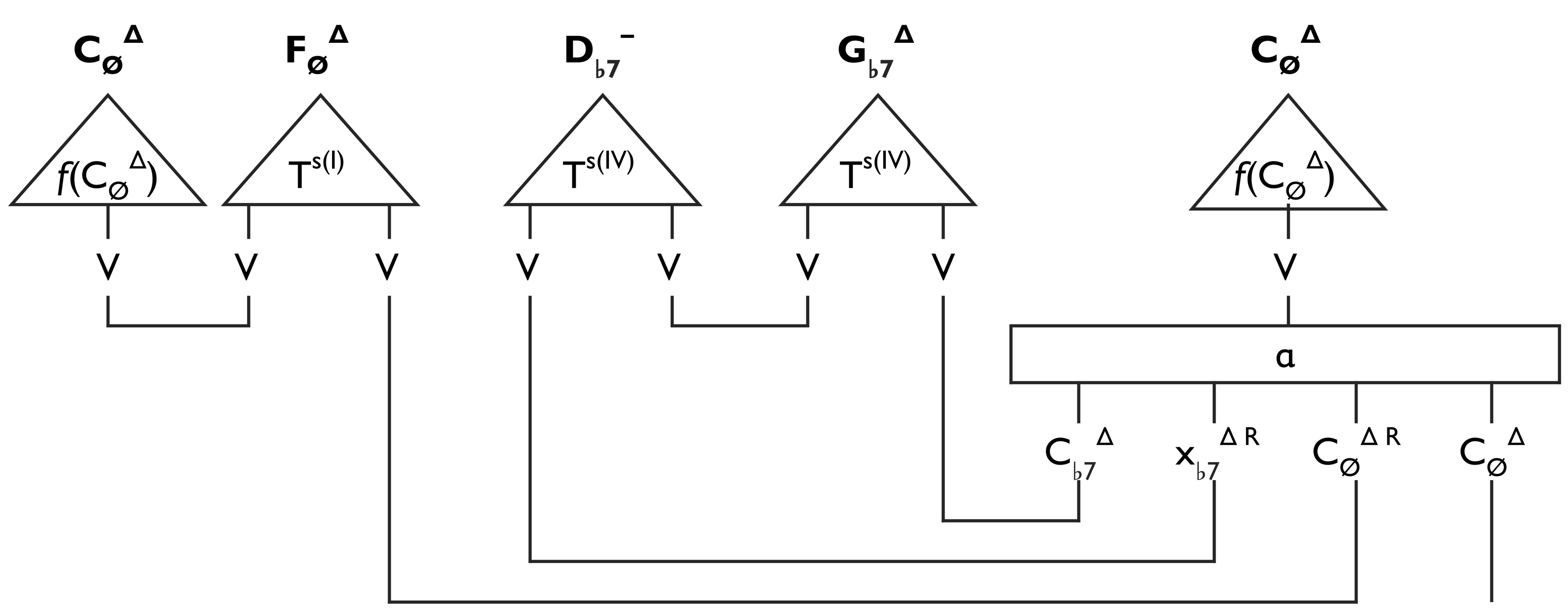
7. Syntactic reduction

Example of syntactic reduction in $\mathcal{J}$ of a simple chord sequence with a short authentic cadence.



8. Semantic interpretation

Semantic interpretation of the example reduction by mapping it to FFreq using functor F.



9. Discussion & conclusion

The syntax of our grammar follows **Steedman's jazz grammar** (2003). We took the first steps towards a meaningful **spectral representation** for semantics of jazz grammars, in line with **cognition-oriented modelling** of Wiggins (2020). The **cadential effect** of jazz can be interpreted as **trajectory** in a space that represents cognitive activity. Syntax and semantics are placed in a **joint category** by following the DisCoCat framework.

Thus, the **compositional structure** specifies the semantics of a chord sequence based on the **harmonic perception of individual chords.** Furthermore, DisCoCat has a wide variety of applications such as measuring the **degree of semantic similarity** (Coecke et al., 2010) of chord sequences and **quantum natural language processing** (Meichanetzidis, 2021).

Vrije Universiteit Brussel

Contact:
luca.iacovella98@gmail.com
geraint.wiggins@vub.be

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