A Hybrid Framework for Chord Analysis: Lambek Calculus and Tonal Pitch Space

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Introduction

Lambek Calculus, a logical system originally developed for categorial grammar, has been proposed as a grammatical framework for analyzing chord sequences in (Bizzarri and Tojo 2024, 2025).

Notation:

- Chords in italics, keys upright.
- Major: uppercase (C = C major), minor: lowercase (a = A minor).
- Degrees: uppercase Roman (I, II, III, ...) for major; lowercase (i, ii, iii, ...) for minor.
- The symbol "/" is taken from categorial grammar (right-associative), not its musical uses (e.g. D/A, I/C).

Lambek Calculus

We translate categorial grammar into a Gentzen-style sequent calculus where $\vdash$ means "derives" and the proofs are Right-hand side restricted to one term (intuitionistic).

$$\frac{}{x \vdash_0 x}\,\text{Chord}\ (\text{Init})$$

$$\frac{\Delta, y, \Sigma \vdash_\delta z \quad \Gamma \vdash_\gamma x}{\Delta, y/x, \Gamma, \Sigma \vdash_{\delta+\gamma+1} z}\,(/_L)$$

$$\frac{\Gamma \vdash_\delta x \quad \Delta, y, \Sigma \vdash_\gamma z}{\Delta, \Gamma, x\backslash y, \Sigma \vdash_{\delta+\gamma+1} z}\,(\backslash_L)$$

$$\frac{\Gamma, x \vdash_\delta y}{\Gamma \vdash_\delta y/x}\,(/_R) \qquad \frac{x, \Gamma \vdash_\delta y}{\Gamma \vdash_\delta x\backslash y}\,(\backslash_R)$$

$$\frac{\alpha : \Gamma \vdash_\delta \Delta}{R_\square \alpha : \square\Gamma \vdash_{\delta+1} \square\Delta}\,(K)$$

Where:

- $\delta, \gamma \in \mathbb{N}$;
- $K_\mathbb{F}$, $R_\mathbb{F}$ are respectively one of the modal functions applied and one of the modal relations;
- $\square$ represents again each modality.

Figure 1: *Rules of the Lambek Calculus for Chord Analysis with the calculus of the depths.*

$$\dfrac{\dfrac{D7 \quad G7}{\mathsf{C{:}\,II} \vdash \mathsf{II} \quad \mathsf{V} \vdash \mathsf{V}}\,(\backslash_L) \qquad \dfrac{}{C_{\mathsf{MA}}{}^7\ \mathsf{I} \vdash \mathsf{I}}}{\mathsf{II}, \mathsf{II}\backslash\mathsf{V}, \mathsf{V}\backslash\mathsf{I} \vdash \mathsf{I}}\,(\backslash_L)$$

Modal Operator

We add axiom (K) with $\square$, we annotate sequents with keys (e.g. C :) and tonal functions ($R_\mathbb{D}$, $R_\mathbb{S}$, $R_\mathbb{P}$, $R_\mathbb{R}$).

$$\frac{\Gamma \vdash x}{\square\Gamma \vdash \square x}\,(K)$$

- $\square$: access to another world (if-clauses, beliefs).
- Kripke semantics: $w \Vdash \square P \iff \forall w'\ (R^w_{w'})\ w' \Vdash P$. Saying $\square P$ holds in the possible world w, if and only if P holds in the all accessible worlds w' from w ($R^w_{w'}$).
- In music: a key is a world. Example: $G \Vdash$ IV means chord C is IV in G.

Multiple Accessibility in Tonal Pitch Space

We model tonal relations as modalities:

$$\mathbb{D} = \text{dominant}, \quad \mathbb{S} = \text{subdominant},$$
$$\mathbb{P} = \text{parallel}, \quad \mathbb{R} = \text{relative}.$$

- Each behaves like $\square$, accessing exactly one other key.
- With 24 tonalities, modality $\leftrightarrow$ key is one-to-one.

$$R_\mathbb{D} R_\mathbb{S} \varphi = R_\mathbb{S} R_\mathbb{D} \varphi = \varphi \tag{1}$$
$$R^{12}_\mathbb{D} \varphi = R^{12}_\mathbb{S} \varphi = \varphi \tag{2}$$
$$R^2_\mathbb{R} \varphi = \varphi,\ R^2_\mathbb{P} \varphi = \varphi \tag{3}$$
$$\begin{cases} R_\mathbb{D} R_\mathbb{P} \varphi = R_\mathbb{P} R_\mathbb{D} \varphi \\ R_\mathbb{S} R_\mathbb{P} \varphi = R_\mathbb{P} R_\mathbb{S} \varphi \\ R_\mathbb{D} R_\mathbb{R} \varphi = R_\mathbb{R} R_\mathbb{D} \varphi \\ R_\mathbb{R} R_\mathbb{S} \varphi = R_\mathbb{S} R_\mathbb{R} \varphi \end{cases} \tag{4}$$

Table 1: *Relations in accessibility*

Main properties:

- $R_\mathbb{D} R_\mathbb{S} \varphi = \varphi$ (dominant of subdominant returns to original key).
- $R^{12}_\mathbb{D} \varphi = \varphi$ (circle of fifths).
- $R^2_\mathbb{R} \varphi = \varphi$, $R^2_\mathbb{P} \varphi = \varphi$.
- Operators commute in pairs, e.g. $R_\mathbb{D} R_\mathbb{P} \varphi = R_\mathbb{P} R_\mathbb{D} \varphi$.

In principle, fewer operators suffice (e.g. $R^{-1}_\mathbb{D} = R_\mathbb{S}$, $R_\mathbb{P} = R^3_\mathbb{S} R_\mathbb{R}$), but we keep all four to highlight musical meaning.

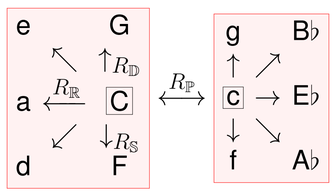


Figure 2: *Accessible keys in the Tonal Pitch Space*

Main relations are the following:

- $\mathbb{D}$: $x \mapsto x + 3 \pmod 7$
- $\mathbb{S}$: $x \mapsto x + 4 \pmod 7$
- $\mathbb{P}$: major $\leftrightarrow$ minor (same tonic)
- $\mathbb{R}$: $M \mapsto m + 5$, $m \mapsto M + 2$ (mod 7)

Only three operators are primitive: $\mathbb{S}$ can be derived from $\mathbb{D}$, $\mathbb{R}$ from $\mathbb{P}$.

$\mathbb{P}$	key, deg_M	=	key, deg_m
$\mathbb{D}$	key, deg_M	=	$\mathbb{D}\{key\}, \{deg + 3 \pmod 7\}_M$
$\mathbb{D}$	key, deg_m	=	$\mathbb{D}\{key\}, \{deg + 3 \pmod 7\}_m$
$\mathbb{S}$	key, deg_M	=	$\mathbb{S}\{key\}, \{deg + 4 \pmod 7\}_M$
$\mathbb{S}$	key, deg_m	=	$\mathbb{S}\{key\}, \{deg + 4 \pmod 7\}_m$
$\mathbb{R}$	key, deg_M	=	$\mathbb{R}\{key\}, \{deg + 5 \pmod 7\}_m$
$\mathbb{R}$	key, deg_m	=	$\mathbb{R}\{key\}, \{deg + 2 \pmod 7\}_M$

A couple of examples:

$$\dfrac{\dfrac{\dfrac{D7 \quad G7}{\mathsf{G{:}\,V} \vdash \mathsf{V} \quad \mathsf{G{:}\,I} \vdash \mathsf{I}}\,(\backslash_L)}{\mathsf{V}, \mathsf{V}\backslash\mathsf{I} \vdash \mathsf{I}}}{\dfrac{R_\mathbb{S}\mathsf{G{:}}\,\mathbb{S}\{\mathsf{V}, \mathsf{V}\backslash\mathsf{I}\} \vdash \mathbb{S}\mathsf{I}}{\mathbb{S}\{\mathsf{V}, \mathsf{V}\backslash\mathsf{I}\} \vdash \mathsf{V}} \qquad \dfrac{}{C_{\mathsf{MA}}{}^7\ \mathsf{I} \vdash \mathsf{I}}}{R_\mathbb{S}\mathsf{G{:}}\,\mathbb{S}\{\mathsf{V}, \mathsf{V}\backslash\mathsf{I}\}, \mathsf{V}\backslash\mathsf{I} \vdash \mathsf{I}}\,(\backslash_L)}\,(\mathsf{K}_\mathbb{S})$$

$$\dfrac{\dfrac{\dfrac{A7^{alt} \quad Dm7}{\mathsf{d{:}\,V} \vdash \mathsf{V} \quad \mathsf{i} \vdash \mathsf{i}}\,(\backslash_L)}{\mathsf{V}, \mathsf{V}\backslash\mathsf{i} \vdash \mathsf{i}}}{R_\mathbb{R}\mathsf{d{:}}\,\mathbb{R}\{\mathsf{V}, \mathsf{V}\backslash\mathsf{i}\} \vdash \mathbb{R}\mathsf{i}}\,(\mathsf{K}_\mathbb{R})}{R_\mathbb{D} R_\mathbb{R}\mathsf{d{:}}\,\mathbb{D}\mathbb{R}\{\mathsf{V}, \mathsf{V}\backslash\mathsf{i}\} \vdash \mathbb{D}\mathbb{R}\mathsf{i}}\,(\mathsf{K}_\mathbb{D})}{\mathsf{VI}, \mathsf{VI}\backslash\mathsf{ii} \vdash \mathsf{ii}} \qquad \dfrac{}{G7\ \mathsf{V} \vdash \mathsf{V}}}{\mathsf{VI}, (\mathsf{VI}\backslash\mathsf{ii})/\mathsf{V}, \mathsf{V} \vdash \mathsf{ii}}\,(/_L)$$

Depth of a Proof

Definition 1 (Depth of a proof)**.** The depth of a proof is the maximum level of nesting in a proof, determined according to the rules in Figure 1.

The concept of depth allows us to more precisely determine how directly a proof represents the harmonic analysis of a given sequence of chords. For example:

$$\dfrac{\dfrac{\dfrac{A7 \quad D7}{\mathsf{C{:}\,VI} \vdash \mathsf{VI} \quad \mathsf{II} \vdash \mathsf{II}}\,(\backslash_L)}{\mathsf{VI}, \mathsf{VI}\backslash\mathsf{II} \vdash \mathsf{II}} \qquad \dfrac{\dfrac{G7 \quad C7}{\mathsf{V} \vdash \mathsf{V} \quad \mathsf{I} \vdash \mathsf{I}}\,(\backslash_L)}{\mathsf{V}, \mathsf{V}\backslash\mathsf{I} \vdash \mathsf{I}}}{\mathsf{VI}, \mathsf{VI}\backslash\mathsf{II}, \mathsf{II}\backslash\mathsf{V}, \mathsf{V}\backslash\mathsf{I} \vdash \mathsf{I}}\,(\backslash_L)$$

Definition 2 (Minimality of a proof)**.** A proof is *minimal* when its depth is lowest among all possible analyses of a sequence of chords.

Definition 3. The minimality of a proof that does not require modal operators is the number of chords minus 1.

Analysis of *All the Things You Are*

All the things you are is a well-known song in the jazz repertoire, as was recorded by many celebrated artists, like John Coltrane, Keith Jarrett or Chet Baker.
The initial tree analyzes a classic vii-ii-V-I progression that resolves to IV, after which there is a change in tonality. The following it's a possibility that we have considered:

$$\dfrac{\dfrac{\vdots \quad \vdots}{\mathsf{vi}, \mathsf{vi}\backslash\mathsf{ii} \vdash_1 \mathsf{ii} \quad \mathsf{V}\backslash\mathsf{I} \vdash_1 \mathsf{I}}\,(\backslash_L) \qquad D\flat_{\mathsf{MA}}{}^7}{\dfrac{\mathsf{vi}, \mathsf{vi}\backslash\mathsf{ii}, \mathsf{ii}\backslash\mathsf{V}, \mathsf{V}\backslash\mathsf{I} \vdash_3 \mathsf{I}}{\mathsf{vi}, \mathsf{vi}\backslash\mathsf{ii}, \mathsf{ii}\backslash\mathsf{V}, \mathsf{V}\backslash\mathsf{I}/\mathsf{IV}, \mathsf{IV} \vdash_4 \mathsf{I}}\,(/_L) \qquad \mathsf{IV} \vdash_0 \mathsf{IV}}\,(/_L)$$

$$\dfrac{\mathsf{G7} \qquad C_{\mathsf{MA}}{}^7}{\mathsf{C{:}\,V} \vdash_0 \mathsf{V} \quad \mathsf{I} \vdash_0 \mathsf{I}}\,(\backslash_L)$$

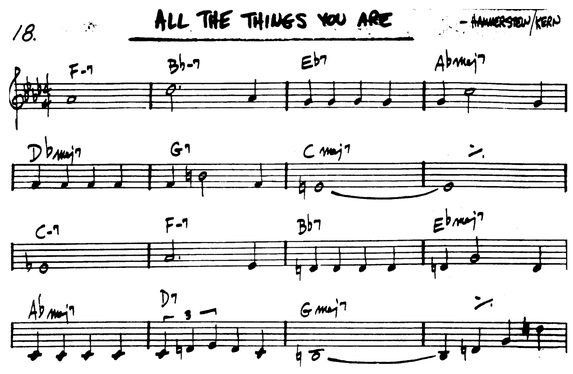
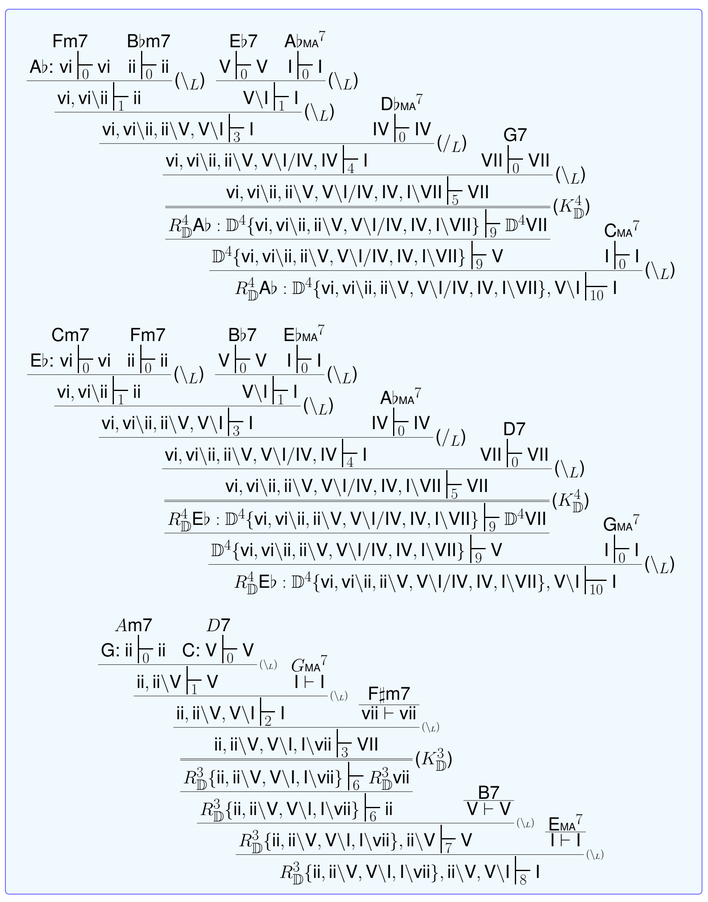


Figure 3: *Section A and B of All the things you are*

Here's the final result:



Ultimately, these differences reflect the main purpose of Lambek Calculus: to serve as a tool for the analyst, who can choose the approach that best fits their analyses.

Conclusions and Future Work

We have applied Lambek Calculus, corresponding to categorial grammar, to the analysis of chord sequences. By introducing the notion of depth, we provided a formal measure of harmonic complexity that clarifies tonal shifts and allows multiple valid interpretations, balancing minimality with explanatory richness.
Future works:

- New theorems will refine the formal properties of the system.
- An automated tool for analysis will make the approach more accessible and efficient.
- Formulas such as $V\backslash I$ can be linked to musical concepts like cadences, bridging symbolic notation and musical meaning.
- Empirical applications on large datasets will further test the usefulness of depth as a parameter, while generative experiments may allow the system not only to analyze but also to produce harmonic progressions with controlled levels of complexity.