The Computational Representation of Octave Equivalence in the Western Staff Notation System

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Abstract

MIPS is a formal language for investigating the mathematical properties of equal-tempered pitch systems and scales within those systems. It is based on four distinct mathematical representations of 'octave equivalence': chroma equivalence, morph equivalence, chromamorph equivalence and genus equivalence. Genus equivalence correctly models the traditional tonal concept of octave equivalence wherein two pitches are considered 'octave equivalent' if and only if they are an integer number of perfect octaves apart.

Futhermore, genus equivalence can be generalized to any pitch system without first having to know 'where the white notes are' in the system.

For each of the four models of octave equivalence there exists a system of definitions and theorems analogous to pitch class set theory.

1 MIPS

MIPS is a formal language devised by the author for investigating the mathematical properties of equal-tempered pitch systems and their associated notational systems.[1] It is fully defined in [8]. MIPS has been implemented as a computer program written in Lisp.

MIPS models the way that pitch information is represented within Western staff notation. In fact, it models a whole class of pitch notation systems that contains the Western staff notation system as one of its members. In this sense, MIPS mathematically models and generalizes the cognitive structure of the pitch representation system used in Western staff notation.

MIPS is based on four representations of 'octave equivalence' including *chroma equivalence* and *genus equivalence*. Chroma equivalence is essentially identical to the concept of 'pitch class equivalence' used by Babbitt ([3]), Forte ([7]), Rahn ([10]), Morris ([9]) and many others. *Genus equivalence* is a new representation invented by the author which provides a correct cognitive model of the traditional tonal concept of 'octave equivalence'. That is, two pitches are genus equivalent if and only if they are an integer number of perfect octaves apart. Genus equivalence can also be generalized to other pitch systems in which scales contain more or less than 7 notes and the octave is divided into more or less than 12 equal intervals.

2 Representing octave equivalence

Chroma equivalence is not a very good cognitive model of the traditional tonal concept of octave equivalence. The three pitches in Figure 1 are 'octave equivalent' in the traditional tonal sense and, of course, they have the same chroma (in this case, 3)—they are therefore chroma equivalent.

However, the two pitches in Figure 2 are also chroma equivalent but they are not 'octave equivalent' in the traditional tonal sense because the in-

[1] MIPS stands for Mathematical Investigation of Pitch Systems

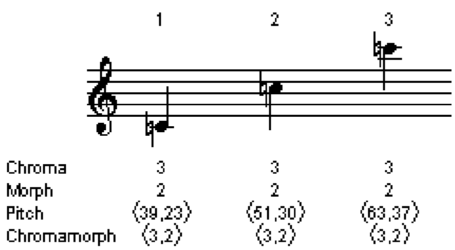


Figure 1: Three pitches that are chroma equivalent and ‘octave equivalent’ in the traditional tonal sense.

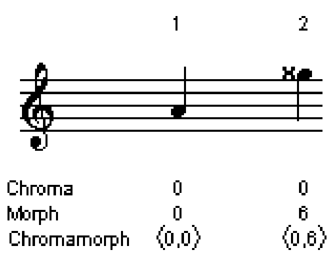


Figure 2: Two pitches that are chroma equivalent but not ‘octave equivalent’ in the traditional tonal sense and not chromamorph equivalent.

terval between them is an augmented seventh and not a perfect octave. So although the sounds produced when the two notes are performed in an equal-tempered system might be psycho-acoustically an octave apart, they are not ‘octave equivalent’ in terms of the cognitive logic of the Western tonal pitch notation system.

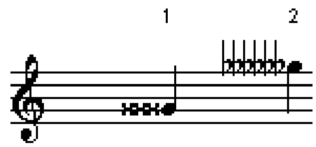
This demonstrates that the concept of pitch class in the sense of Forte ([7]), Rahn ([10]) and others, is not a correct cognitive model of ‘octave equivalence’ in the Western tonal pitch notation system.

There have been a number of attempts to produce better models of the traditional tonal concept of ‘octave equivalence’. For example, Brinkman ([4], 128) and Agmon ([1], 11; [2], 44) use a representation of octave equivalence that Brinkman calls a ‘binomial representation’. MIPS incorporates a representation called *chromamorph* which is essentially identical to Brinkman’s ‘binomial representation’. A chromamorph is an ordered pair in which the first number represents the chroma and the second number (which in MIPS is called *morph* and which Brinkman calls ‘name class’ ([4], 124–126)) represents the letter-name of the note. So, in the Western system, this second number—the morph—will range over the integers 0–6. But in a system that uses five-note scales, the morph would take an integer value between 0 and 4.

If two notes that have the same chromamorph are defined to be *chromamorph equivalent* then it can be seen from Figure 2 that chromamorph equivalence is a better model of the Western tonal concept of ‘octave equivalence’ than ‘chroma equivalence’—two notes an augmented seventh apart are not chromamorph equivalent just as they are not ‘octave equivalent’ in the traditional tonal sense.

However, the two notes in Figure 3 *are* chromamorph equivalent but they are certainly *not* ‘octave equivalent’ in the traditional Western tonal sense—the interval between them is a ‘12×diminished octave’. This demonstrates that chromamorph equivalence is not a correct cognitive model of the traditional concept of Western tonal octave equivalence.

In the traditional Western tonal pitch-naming system, a note has a letter-name (A to G), an inflection $(\ldots, \flat\flat, \flat, \natural, \sharp, \sharp\sharp, \ldots)$ and an octave number (for ex-



Chroma	4	4
Morph	6	6
Pitch	⟨52,27⟩	⟨52,34⟩
Chromamorph	⟨4,6⟩	⟨4,6⟩
Genus	⟨16,6⟩	⟨4,6⟩
Old genus	⟨6,6⟩	⟨6,-6⟩

Figure 3: Two pitches that are chromamorph equivalent but not octave equivalent in the traditional tonal sense.

ample, middle C—$C\natural_4$—has an octave number of 4 and the C above middle C ($C\natural_5$) has an octave number of 5). This naming system derives from the staff notation system which has evolved over the past 400 years to be a highly effective means of notating Western tonal music. To this extent, the pitch-naming system correctly models the Western tonal pitch system. And if the octave number of a pitch-name is omitted (for example, $C\natural_4$ becomes $C\natural$), the result is a correct representation of 'octave equivalence' within the Western tonal system.

So, if one wishes to find a correct mathematical representation of the traditional tonal concept of octave equivalence, one strategy might be to base a numerical representation on the traditional pitch-naming system. A number of researchers have done this (including Cambouropoulos ([5], 233; [6], 49). In this system, the letter-name (A to G) is represented by a number between 0 and 6 and the inflection (or 'modifier-accidental' as Cambouropoulos calls it) is represented by an integer (0 corresponds to $\natural$, 1 corresponds to $\sharp$, -1 corresponds to $\flat$ and so on).

The row labelled 'Old genus' in Figure 3 shows that this representation correctly captures the fact that the two notes are not 'octave equivalent' in the traditional sense. So this simple numeric representation of the Western tonal pitch naming system provides a correct cognitive model of the traditional concept of 'octave equivalence' within that system.

However, one of the motivations behind the development of MIPS was to produce a system that would allow one to examine the special mathematical properties of the Western tonal scales—and particularly the diatonic scale—and then go on to determine if scales with similar properties exist in systems where the octave is divided into more or less than 12 equal divisions. In other words, it should be possible to use MIPS to find out 'where the white notes should be' in other equal-tempered pitch systems. But unfortunately, it is not possible to generalize a representation such as Cambouropoulos' to other equal-tempered pitch systems without first knowing 'where the white notes are' because one first has to know which pitch classes correspond to the naturals.

3 Genus

It turns out, however, that it *is* possible to produce a correct mathematical model of traditional tonal 'octave equivalence' that *is* generalizable to any equal-tempered pitch system and does *not* require one first to know 'where the white notes are' in that pitch system.

In MIPS, this model of octave equivalence is called *genus equivalence*: two pitches are genus equivalent if and only if they have the same *genus*. A genus is an ordered pair rather like a chromamorph. As in a chromamorph, the second number is a morph and represents the letter-name (see Figure 3). However, the first member of a genus is not a chroma but a *chromatic genus* which is not quite the same as chroma. Unfortunately the fact that chromatic genus is 'not quite' chroma means that the whole theory surrounding the genus representation—the theory that defines, for example, how to transpose and invert genus sets, find their interval vectors and so on—is rather more involved than the pitch class set theory of Babbitt, Forte and Rahn.

4 Summary

In summary, MIPS is a formal language for investigating the mathematical properties of equal-tempered pitch systems and scales within those systems. It is based on four distinct mathematical representations of 'octave equivalence': chroma equivalence, morph equivalence, chromamorph equivalence and genus equivalence. Genus equivalence correctly models the traditional tonal concept of octave equivalence wherein two pitches are considered 'octave equivalent' if and only if they are an integer number of octaves apart.

Futhermore, genus equivalence can be generalized to any pitch system without first having to know 'where the white notes are' in the system.

For each of the four models of octave equivalence there exists a system of definitions and theorems analogous to pitch class set theory. For example, one component of MIPS deals with chromamorph and states a number of definitions and theorems concerning chromamorph that describe, for example, how to transpose, invert and find the interval vectors of chromamorph sets. Another component does the same thing for genus sets and so on. In addition, for each model of octave equivalence there exists within MIPS a system for representing relations using digraphs. These systems can be used to show that the tonal scales possess certain unique graph-theoretical properties.

A full specification of MIPS including a formal definition of the concept of genus can be found in [8]. Enquiries relating to the material presented in this article should be addressed to the author whose e-mail address is as follows:

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